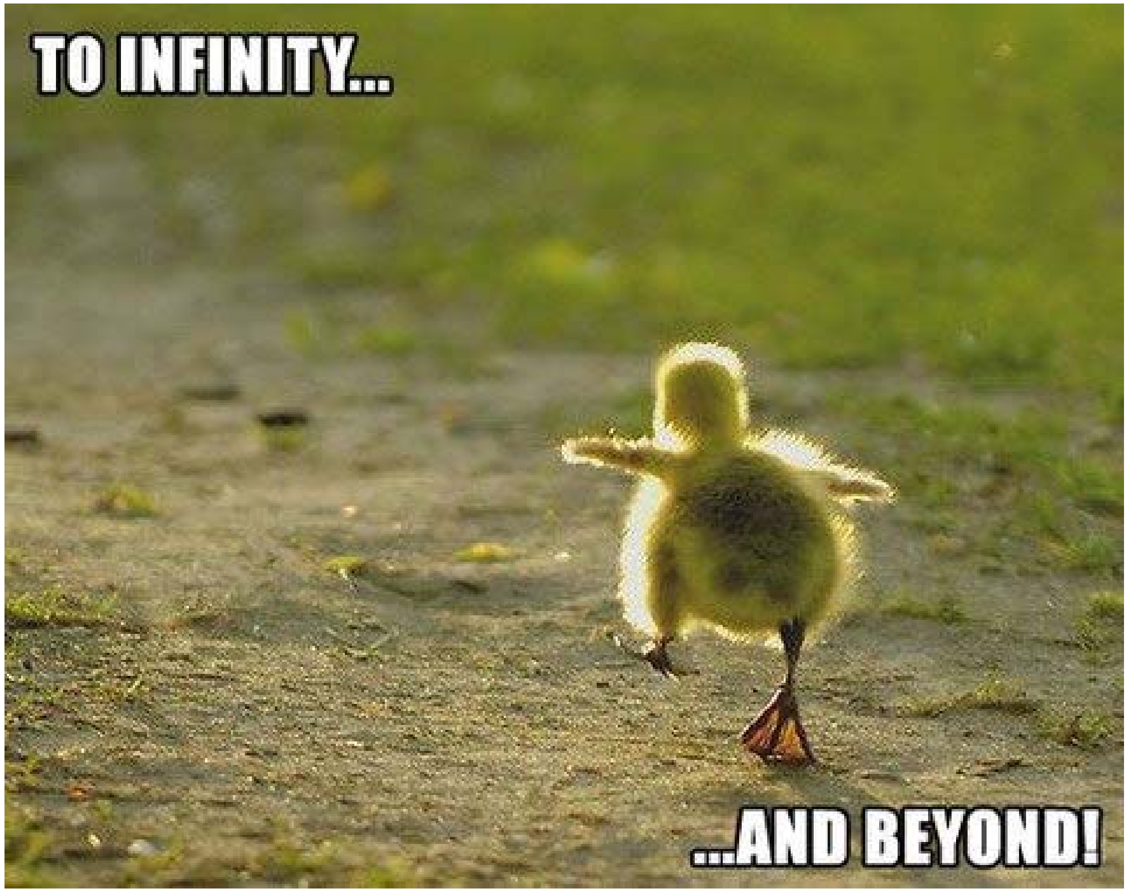
Countability



Discrete Structures (CS 173) Lecture B

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Based on slides by Derek Hoiem, University of Illinois

Final exam times/rooms

Wednesday, **Dec 15, 7-8:45pm**

Siebel Center 1404: Students in **Lecture BL2** with last names **A to L**

DCL 1320: Students in **Lecture BL2** with last names **M to Z**

Tell me about any conflicts asap. Note there is no specific conflict exam --- most conflicts should be resolved by other classes.

http://admin.illinois.edu/policy/code/article3_part2_3-201.html

Make-up Exams

- If you are taking a make-up examlet:
  - please inform your TA right away.
  - It will be administered 8:45 to 10pm on Thursday 12/15/2016 (right after the Final). Simply stay in the room.

Today's class: Countability and Computability

- Are some infinite sets bigger than other infinite sets?
  - How big are: Natural numbers, integers, rationals, reals, powerset of naturals
- What does it mean for a set to be "countable"?
- How do we prove that a set is or is not countable?
- Is every function computable?
- Can we have an algorithm to decide if a program will terminate?

Are there more integers than natural numbers?

Are there more integers than natural numbers?

$|A| = |B|$ iff there is a bijection from A to B.

Are there more rational numbers than integers?

$|A| = |B|$ if there is a bijection from A to B.

Are there more rational numbers than integers?

Cantor Schroeder Bernstein Theorem: $|A| = |B|$ if there exist one-to-one (injective) functions $f: A \to B$ and $g: B \to A$

Countability

A set is **countably infinite** if it has the same size as the set of natural numbers.

A set is **countable** if it is finite or countably infinite.

Countable sets:

– Any subset of a countable set is countable

– The Cartesian product of finitely many countable sets is countable

– A union of countably many countable sets is countable

What sets are not countable?

Power set of natural numbers is not countable

Proof by contradiction and "diagonalization" (Cantor)

- Write a subset of natural numbers as an infinite-length binary string
- Suppose there is a complete list of such strings (could be infinitely many of them)
- Can construct a new string that is different from all infinity of them!

Cardinality of Powersets

- The powerset of a set is always larger than the set.

Reals are not countable

Similar to proof for powerset of naturals

- Restrict ourselves to a subset of reals: those between 0 and 1
- Each real number is a decimal followed by a potentially infinite number of 0 or 1 numerals
- Can construct a new real number by diagonalization

Which is bigger: set of real numbers or power set of natural numbers?

Diagonalization

Why doesn't the diagonalization technique work for disproving that all pairs of natural numbers is countable?

Hilbert's Paradox of the Grand Hotel

Suppose the Grand Hotel has a countably infinite number of rooms, which are all occupied.

– How can the hotel accommodate one more person without making anyone leave?

– How can the hotel accommodate a countably infinite number of new people?

The Continuum Hypothesis (CH)

Is there any set that is larger than the set of natural numbers but smaller than the set of real numbers?

- Posed by Cantor in 1880s
- In 1931, Gödel showed that there are true statements that can't be proven true in any axiom system and, later, that the negation of the CH is one of them
- Thus, CH is either true or it's false but can't be proven false from the axioms of set theory
- Later, Paul Cohen proved that the continuum hypothesis cannot be proved from the axioms of set theory
- Thus, no logical conflict can occur from asserting the CH or its negation

Details: http://www.people.vcu.edu/~rhammack/BookOfProof/Cardinality.pdf

Summary: compare set sizes

Integer vs. Natural

Natural vs. Real

Powers of 4 vs. Integers

Real vs. Rational

Irrational vs. Rational

Powerset(Natural) vs. Real

Powerset(Real) vs. Real

Uncomputability

- A computer program is just a string (finite series of characters), so it is countable
- But the set of functions is uncountable (e.g., functions that map integers to integers)
- So there are more functions than programs – some functions cannot be computed by any program
- Can we prove that a program will halt?

Example: not all real numbers are computable

3/4

$\sqrt{2}$

http://keisan.casio.com/calculator

Halting problem: Is there a general purpose algorithm that can determine whether a program will terminate?

Example: Suppose you have written a machine learning algorithm that has been running for days. Will it ever stop on its own?

Potential solutions to halting problem

- Run program for a really long time and see if it stops
  - May keep running forever
- Analyze code to see if there are infinite loops
  - Sometimes we can tell : $\text{While true do x} := \text{x}$
- Check if loop exit conditions become closer to being met over time
  - Can use induction

Things to remember

- Some infinite sets are bigger than others
- We can compare sizes of infinite sets using bijections or one-to-one functions in each direction
- A “countable” set is the same size (or smaller) than the natural numbers